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ON A METHOD OF SOLVING MECHANICS PROBLEMS FOR DOMAIN WITH SLITS OR THIN INCLUSIONS

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A method is developed to solve problems of the mechanics of a continuous medium and mathematical physics for domains with slits (cracks) or thin inclusions based, firstly, on an integral transform in a variable intersecting the slit or inclusion, secondly, on the formulation of the boundary value problem under investigation in the form of a system of first order differential equations. The method is illustrated by specific mechanics problems.

R. V. Serebrianyi [1] apparently first used the idea of an integral (Fourier) transform on a line intersecting a slit in solving the problem of the bending of an infinite plate, hinge-slit along an infinite line.

1. Let us consider the boundary value problem for the equation

$$r_1(\xi)\frac{\partial}{\partial\xi}\left[p_1(\xi)\frac{\partial u}{\partial\xi}\right]+r_2(\eta)\frac{\partial}{\partial\eta}\left[p_2(\eta)\frac{\partial u}{\partial\eta}\right]-q_1(\xi)u-q_2(\eta)u=0 \qquad (1.1)$$

in the domain shown in Fig. 1, with boundary conditions of general type on the coordinate lines

$$\xi=\xi_0,\ \xi=\xi_*;\ \eta=\eta_1,\ \eta=\eta_4.$$

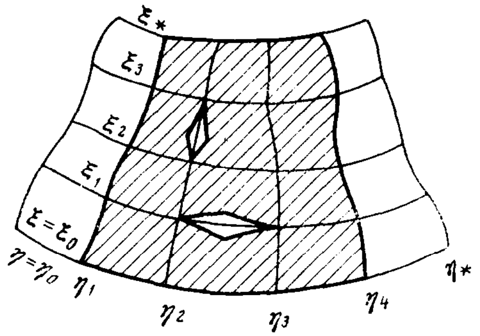

Fig. 1

We assume that there is a slit or a thin inclusion on the coordinate line $\xi=\xi_1$ for $\eta_2<\eta<\eta_3$, i.e., a line on which the required function becomes discontinuous (but its normal derivative is specified).

$$u|_{\xi=\xi_1-0}-u|_{\xi=\xi_1+0}=\chi(\eta),\quad \frac{\partial u}{\partial\xi}\bigg|_{\xi_1-0}=\frac{\partial u}{\partial\xi}\bigg|_{\xi_1+0}=g(\eta)\quad (\eta_2\leqslant\eta\leqslant\eta_3) \qquad (1.2)$$

or the normal derivative becomes discontinuous (but values of the function are specified, i.e.,

$$u|_{\xi_1-0}=u|_{\xi_1+0}=g(\eta),\quad \frac{\partial u}{\partial\xi}\bigg|_{\xi_1-0}-\frac{\partial u}{\partial\xi}\bigg|_{\xi_1+0}=\chi(\eta)\quad(\eta_2\leqslant\eta\leqslant\eta_3) \qquad (1.3)$$

Here $\chi(\eta)$ is an unknown function equal to zero on the continuation of the slit or inclusion, and $g(\eta)$ is a specified function.

The usual means (for example [2]) of solving such problems by the integral transform method is based on partitioning the domain under investigation (Fig. 1) into two parts by a coordinate line $\xi = \xi_1$ containing the slit. Subsequent connection of the solutions for these domains with conditions (1.2) or (1.3) taken into account results in dual integral equations.

Let us use another way of solution which does not require partitioning the initial domain into parts and is based on the presence of a transform in the variable ξ in the interval (ξ_0, ξ_*) expressed in the general case by the formulas

$$u_\lambda(\eta) = \int_{\xi_0}^{\xi_*} u(\xi, \eta) K(\xi, \lambda) r_1^{-1}(\xi) d\xi, \quad u(\xi, \eta) = \int_l R(\xi, \lambda) u_\lambda(\eta) d\sigma(\lambda) \tag{1.4}$$

(l can represent a certain line in the plane of the complex variable λ).

To realize the method developed here, let us multiply both sides of (1.1) by $r_1^{-1}(\xi)K(\xi, \lambda)$ and let us integrate by parts separately in the intervals (ξ_0, ξ_1) and (ξ_1, ξ_*). Use of a differential equation and boundary conditions of the Sturm–Liouville problem, whose solution [3] is the kernel $K(\xi, \lambda)$, and also of the notation (1.4), permits reduction of (1.1) to the following:

$$r_2 \frac{d}{d\eta}\left(p_2 \frac{d}{d\eta} u_\lambda\right) - (q_2 + \lambda) u_\lambda = n(\lambda)\left[u|_{\xi_1-0} - u|_{\xi_1+0}\right] - \tag{1.5}$$
$$l(\lambda)\left[\frac{\partial u}{\partial \xi}\Big|_{\xi_1-0} - \frac{\partial u}{\partial \xi}\Big|_{\xi_1+0}\right] \quad (\eta_1 < \eta < \eta_4)$$
$$n(\lambda) = \left[p_1 \frac{\partial K}{\partial \xi}\right]_{\xi=\xi_1}, \quad l(\lambda) = [p_1 K]_{\xi=\xi_1}$$

For example, let the condition (1.2) be realized on the slit (inclusion), then we can write in place of (1.5)

$$r_2 \frac{d}{d\eta}\left(p_2 \frac{d}{d\eta} u_\lambda\right) - (q_2 + \lambda) u_\lambda = n(\lambda)\chi(\eta) \quad (\eta_1 < \eta < \eta_4) \tag{1.6}$$

Boundary conditions transformed along the boundary lines $\eta = \eta_1$ and $\eta = \eta_4$ having the following form in the general case [4]

$$U_j[u_\lambda] = \gamma_j \quad (j = 0, 1) \tag{1.7}$$

should still be added to the equation obtained.

If the Green's function $G(\eta, \sigma)$ of the semi-homogeneous $(\gamma_j = 0, j = 0,1)$ boundary value problem (1.6), (1.7) is constructed and the basis system of functions $\psi_j(\eta)$, $j = 0,1$ is known which satisfies the homogeneous differential equation (1.6) and the boundary conditions

$$U_j[\psi_k] = \delta_{jk} \quad (j, k = 0, 1) \tag{1.8}$$

then the solution of the boundary value problem (1.6), (1.7) will have the form [5]

$$u_\lambda(\eta) = \int_{\eta_2}^{\eta_3} G(\eta, \sigma) n(\lambda) \chi(\sigma) d\sigma + \sum_{j=0}^{1} \gamma_j \psi_j(\eta) \tag{1.9}$$

Furthermore, using the inversion formula (1.4), we find the original $u(\xi, \eta)$

for the transform (1.9). Substituting its derivative with respect to ξ into (1.2), we arrive at an integral equation to determine $\chi(\eta)$. As a rule, it turns out to be singular.

To extract the singular part explicitly beforehand, it is recommended to take the Green's function in the following form:

$$G(\eta, \sigma) = \Phi(\eta, \sigma) - \sum_{j=0}^{1} \psi_j(\eta) U_j[\Phi] \tag{1.10}$$

where $\Phi(\eta, \sigma)$ is understood to be a fundamental function which is used to express the solution $u(\eta)$ of (1.6) with an arbitrary right side $f(\sigma)$ by the formula

$$u(\eta) = \int_{\eta_0}^{\eta_*} \Phi(\eta, \sigma) f(\sigma)\, d\sigma \tag{1.11}$$

If (1.8) is taken into account with the governing properties of the Green's function [4,5], it can be seen that (1.10) actually defines the Green's function of the boundary value problem (1.6), (1.7).

Note. It is precisely the first term in (1.10) which determines the singular part in the kernel of the integral equation mentioned. The formula (1.10) presented here for the Green's function is apparently new. It remains valid even for differential equations of arbitrary order (only the quantity of terms under the summation sign changes).

It is simplest to construct the fundamental function $\Phi(\eta, \sigma)$ by using the integral transform

$$\int_{\eta_0}^{\eta_*} K_2(\eta, \mu)\, r_2^{-1}(\eta) f(\eta)\, d\eta = f_\mu, \quad f(\eta) = \int_{l_2} R_2(\eta, \mu) f_\mu d\sigma_2(\mu) \tag{1.12}$$

whose kernel is the solution of the Sturm—Liouville problem for the equation

$$r_2(\eta) \frac{d}{d\eta}\left[p_2(\eta) \frac{d}{d\eta} K_2\right] - q_2 K_2 = -\mu K_2 \quad (\eta_0 < \eta < \eta_*) \tag{1.13}$$

In the case of (1.6) with constant coefficients, the fundamental function constructed in such a way becomes dependent on the difference between the arguments and it turns out to be convenient to take one of the following

$$\Phi(\eta, \sigma) = \Phi(\eta - \sigma) \pm \Phi(\eta + \sigma) \tag{1.14}$$

as $\Phi(\eta, \sigma)$ in (1.10).

2. Let us realize the scheme elucidated in an example of the following problem. There is a thin stiff inclusion in the form of a strip $0 \leqslant x \leqslant a$ $(a < 1)$; $-\infty < z < \infty$ in the $y = 0$ plane in an elastic layer $(0 \leqslant x \leqslant 1,\ -\infty < y, z < \infty)$ with the clamped edge $x = 1$. Find the stress field if a uniformly distributed shearing (i.e., acting along the z-axis) load is applied to the outer edge of the inclusion mentioned. This antiplane problem is equivalent to the boundary value problem

$$\Delta w = 0 \quad (-\infty < y < \infty,\ 0 < x < 1) \tag{2.1}$$

$$\left.\frac{\partial w}{\partial x}\right|_{x=0} = 0, \quad w|_{x=1} = 0 \quad \left(\tau_{xz} = \mu\frac{\partial w}{\partial x}, \ \tau_{yz} = \mu\frac{\partial w}{\partial y} \ \mu\right)$$

with compliance with the following conditions on the inclusion (μ is the shear modulus) representing the analog of the conditions (1.3)

$$\tau_{yz}|_{y=-0} - \tau_{yz}|_{y=+0} = \mu\left[\left.\frac{\partial w}{\partial y}\right|_{-0} - \left.\frac{\partial w}{\partial y}\right|_{+0}\right] = \chi(x) \tag{2.2}$$

$$\left.\frac{\partial w}{\partial x}\right|_{y=0} = 0 \quad (0 \leqslant x \leqslant a, \ \chi(x) \equiv 0, \ x \notin (0, a))$$

Applying a Fourier transform in y to the Laplace equation from (2.1) with a partition (according to the scheme in Sect. 1) of the section of integration into two $(-\infty, -0)$ $(+0, \infty)$ and (2.2) taken into account, we arrive at the following analog of the boundary value problem (1.6), (1.7):

$$\frac{d^2 w_\beta(x)}{dx} - \beta^2 w_\beta(x) = -\frac{\chi(x)}{\mu} \quad (0 < x < 1) \tag{2.3}$$

$$\left.\frac{dw_\beta(x)}{dx}\right|_{x=0} = w_\beta(1) = 0 \quad \left(w_\beta(x) = \int_{-\infty}^{\infty} e^{i\beta y} w(x, y)\, dy\right)$$

We easily find the basis system of functions satisfying the boundarv conditions (1.8)

$$\psi_0(x) = -\frac{\operatorname{sh}\beta(1-x)}{\beta \operatorname{ch}\beta}, \quad \psi_1(x) = \frac{\operatorname{ch}\beta x}{\operatorname{ch}\beta} \tag{2.4}$$

By using the Fourier transform in y (performing the role of the transformation (1.12) here) applied to (2.3), we determine the fundamental function

$$\Phi(x-\xi) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\alpha(x-\xi)}}{\alpha^2 - \beta^2}\, d\alpha = -\frac{e^{-|\beta|\,|x-\xi|}}{2|\beta|} \tag{2.5}$$

With the first espression from (1.14) substituted and (2.4) and (2.5) taken into account, (1.10) results in the following expression for the Green's function of the boundary value problem (2.3):

$$-G(x, \xi) = \frac{e^{-|\beta|\,|x-\xi|} + e^{-|\beta|\,|x+\xi|}}{2|\beta|} - \frac{\operatorname{ch}\beta x \operatorname{ch}\beta\xi}{e^{|\beta|}\,|\beta| \operatorname{ch}\beta} \tag{2.6}$$

Using it, we find $dw_\beta(x) / dx$.

Subsequent utilization of the inversion formula for the Fourier transform and evaluation of the known integrals in the transformation parameter β permit obtaining an expression for $\partial w / \partial x$. Substituting it into (2.2) results in the following integral equation

$$\int_{-a}^{a} \operatorname{cosec}\frac{\pi(x-\xi)}{2}\chi(\xi)\, d\xi = 0 \quad (|x| \leqslant a)$$

to determine the function $\chi(\xi)$ (continued evenly to negative values of the argument).

The equation obtained admits of a simple exact solution containing an arbitrary constant. The value of this latter can be determined from the equilibrium condition for the inclusion.

3. The scheme elucidated in Sect. 1 is generalized in an evident manner to the case of a finite number of inclusions (or slits) located on the coordinate lines $\xi = \text{const}$ (there will be a system instead of just one integral equation). It can also be extended to the case of more general conditions on the inclusions than (1.2) or (1.3).

In principle, the presence of inclusions on both the lines $\xi = \text{const}$ and on the lines $\eta = \text{const}$ is more complex.

In addition to inclusion (slit) on the line $\xi = \xi_1$ (Fig. 1) depicted in Sect. 1, there is still an inclusion (slit) on the line $\eta = \eta_2$. For definiteness, let us consider that conditions of the type (1.2) are realized thereon, i.e.,

$$u|_{\eta_2-0} - u|_{\eta_2+0} = \psi(\xi) \quad (\psi(\xi) \equiv 0,\ \xi \notin (\xi_2, \xi_3)) \tag{3.1}$$

$$\frac{\partial u}{\partial \eta}\Big|_{\eta_2-0} = \frac{\partial u}{\partial \eta}\Big|_{\eta_2+0} = h(\xi) \quad (\xi_2 \leqslant \xi \leqslant \xi_3)$$

Applying the transformation (1.4) to the first of these conditions yields

$$u_\lambda|_{\eta_2-0} - u_\lambda|_{\eta_2+0} = \psi_\lambda \quad \left(\psi_\lambda = \int_{\xi_2}^{\xi_3} r_1^{-1}(\xi) K(\xi, \lambda) \psi(\xi) d\xi\right) \tag{3.2}$$

This condition now needs to be appended to the boundary value problem (1.6), (1.7). In order to satisfy it, we first find the auxiliary function $u_\lambda^*(\eta)$, which should have the jump (3.2) to satisfy the differential equation (1.6) in the intervals (η_0, η_2) and (η_2, η_*), where $\eta_0 \leqslant \eta_1$, $\eta_* \geqslant \eta_4$, and the boundary conditions on the Sturm — Liouville problem for the kernel of the integral transform (1.12) as well.

Such a function is easily found by using this transformation applied to (1.6) in the interval (η_0, η_*) partitioned into two: $(\eta_0, \eta_2 - 0)$ and $(\eta_2 + 0, \eta_*)$ and taking account of the jump (3.2). It will have the form

$$u_\lambda^*(\eta) = -\int_{l_2} \frac{n(\lambda)\chi_\mu + n_2(\mu)\psi_\lambda}{\lambda+\mu} R_2(\eta, \mu)\, d\sigma_2(\mu) \tag{3.3}$$

$$(n_2(\mu) = [p_2(\eta)\partial K_2 / \partial\eta]_{\eta=\eta_2})$$

Now, if values of the functionals contained in (1.7)

$$U_j[u_\lambda^*] = \gamma_j^* \tag{3.4}$$

are found and the basis system of functions satisfying the condition (1.8) is available, it can be seen that the solution of the boundary value problem (1.6), (1.7) in the presence of the jump (3.2) will have the form

$$u_\lambda(\eta) = u_\lambda^*(\eta) + \sum_{j=0}^{1} (\gamma_j - \gamma_j^*)\psi_j(\eta) \tag{3.5}$$

Subsequent utilization of the inversion formula from (1.4) permits finding the function $u(\xi, \eta)$ and its derivatives. Realization of the second conditions from (1.2) and (3.1) on the inclusions (slits) will result in a system of integral equations in the required functions $\chi(\xi)$ and $\psi(\eta)$.

Let us illustrate the scheme elucidated in the problem selected in Sect. 2 by adding the presence of a crack on the segment $x = a, \; -b \leqslant y \leqslant b$ with the conditions

$$w(a-0, y) - w(a+0, y) = \psi(y) \tag{3.6}$$
$$\left.\frac{\partial w}{\partial x}\right|_{a-0} = \left.\frac{\partial w}{\partial x}\right|_{a+0} = 0 \quad (|y| < b)$$

to it, which will result in complicating the boundary value problem (2.3) because of compliance with the condition

$$w_\beta(a-0) - w_\beta(a+0) = \psi_\beta \tag{3.7}$$

obtained by a Fourier transformation of the first conditions (3.6) relative to y.

Let us take the Fourier cosine transform

$$\int_0^\infty \cos\alpha x f(x)\,dx = f_\alpha, \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos\alpha x f_\alpha\,d\alpha \tag{3.8}$$

as the analog of the integral transform (1.12).

Applying it to the differential equation (2.3) with the partition of the interval $(0, \infty)$ into $(0, a-0), (a+0, \infty)$ and the jump (3.7) taken into account, we find the analog of the function (3.3)

$$w_\beta^*(x) = \frac{2}{\pi}\int_0^\infty \frac{\alpha\sin\alpha a\psi_\beta + \mu^{-1}\chi_\alpha}{\alpha^2+\beta^2}\cos\alpha x\,d\alpha \tag{3.9}$$

In the case selected, the values of the functionals from (3.4) have the form

$$\gamma_0^* = \left.\frac{dw^*_\beta}{dx}\right|_{x=0} = 0, \quad \gamma_1^* = w_\beta^*(1) \quad (\gamma_j = 0) \tag{3.10}$$

and the basis system of functions is determined by (2.4). Hence, according to (3.5) the solution of the boundary value problem (2.3) in the presence of the jump (3.7) will be

$$w_\beta(x) = w_\beta^*(x) - \gamma_1^*\,\mathrm{ch}\,\beta x\,\mathrm{sech}\,\beta \tag{3.11}$$

Using the inversion formula for the Fourier transform, we hence find $w(x, y)$. After evaluation of the known integrals in the parameter β we arrive at the following formula

$$\frac{\partial w}{\partial x} = -\frac{\pi}{4\mu}\int_{-a}^{a}\frac{\sin\frac{1}{2}\pi(x-\xi)\,\mathrm{ch}\,\frac{1}{2}\pi y\chi(\xi)}{\mathrm{sh}^2\frac{1}{2}\pi y + \sin^2\frac{1}{2}\pi(x-\xi)}\,d\xi - \tag{3.12}$$
$$\frac{\partial}{\partial x}\int_{-b}^{b}\left[\mathrm{arctg}\,\frac{\mathrm{sh}\,\frac{1}{2}\pi(\eta-y)}{\sin\frac{1}{2}\pi(a-x)} + \mathrm{arc\,tg}\,\frac{\mathrm{sh}\,\frac{1}{2}\pi(\eta-y)}{\sin\frac{1}{2}\pi(a+x)}\right]\frac{\psi'(\eta)}{\pi}\,d\eta$$

Here the function $\chi(x)$ is continued evenly to negative values of the argument. while the derivative of the function $\psi(y)$ appears because of integration by parts. Using the formula obtained, we realize the remaining condition on the inclusion (2.2) and on the crack (3.6). We consequently arrive at the following system of singular integral equations

$$\frac{1}{2\mu}\int_{-a}^{a}\frac{\chi(\xi)\,d\xi}{\sin{}^1/_2\pi(\xi-x)}-\int_{-b}^{b}[s(x,\eta)-s(-x,\eta)]\,\psi'(\eta)\,d\eta=0\quad(|x|\leqslant a)\qquad(3.13)$$

$$\int_{-b}^{b}\left[\frac{1}{\operatorname{sh}{}^1/_2\pi(\eta-y)}-s(a,\eta-y)\right]\psi'(\eta)\,d\eta+$$

$$\frac{1}{2\mu}\int_{-a}^{a}c(y,\xi)\,\chi(\xi)\,d\xi=0\quad(|y|\leqslant b)$$

$$s(x,y)=\frac{\operatorname{sh}{}^1/_2\pi y\cos{}^1/_2\pi(a+x)}{\operatorname{sh}^2{}^1/_2\pi y+\sin^2{}^1/_2\pi(a+x)},\quad c(y,\xi)=\frac{\operatorname{ch}{}^1/_2\pi y\sin{}^1/_2\pi(a-\xi)}{\operatorname{sh}^2{}^1/_2\pi y+\sin^2{}^1/_2\pi(a-\xi)}$$

Let us note that the derivative of the expansion of the crack $\psi'(y)$ can be found from this system. However, the stress intensity factor [6] is of greatest interest. In the case of plane problems, the relation of this factor to the factor of the singularity in the derivative of the crack expansion is established by analyzing complex potentials [6]. An analogous relation can be proposed in the general case also, if the problem is stated in the form of a singular equation in the derivative of the crack expansion. To do this, the known behavior of Cauchy type integrals given on a segment as the variable approaches the ends of integration from an outer and inner point of the mentioned segment should be used [7].

4. In examining the more complex boundary value problems of mathematical physics and the mechanics of continuous media, we must deal with not one second order equations but with a system of such equations or with one equation but of an order higher than the second (elasticity theory, theory of plate and shell bending, etc.).

To solve such problems with slits (cracks) or thin inclusions by the method elucidated above, the differential equation of the boundary value problems under investigation should be written as a complete system of first order differential equations, which permits writing most simply and realizing the condition on the slit or inclusion.

Let us illustrate this in a plate bending problem. A rectangular $(a_0 \leqslant x \leqslant a_1,$ $0 \leqslant y \leqslant b,\ a_1 - a_0 > b)$ plate, hinge-supported along the edges $y = 0$, $y = b$ is subjected to the effect of a normal load $q(x, y) = q$ distributed normally over the whole domain mentioned. When this load reaches the specific quantity $q = q_p$ at the center of the plate, the normal stress σ_y reaches the yield point σ_p (or $M_y = M_p$). A further increase in the load $q > q_p$ under the assumption of no hardening of the material can result in the occurrence of a linear plastic hinge on the segment $c_0 < x < c_1$ on which the condition

$$M_y\big|_{y=l} = M_p\quad(c_0 \leqslant x \leqslant c_1,\ l = {}^1/_2 b)\qquad(4.1)$$

will be satisfied. (Such a hinge definitely occurs if there is a shallow crack on the segment mentioned).

It is required to clarify the stress redistribution in the plate upon the occurrence of the plastic hinge mentioned.

To solve the problem posed by the method developed here, let us start from the complete system of first order equations in the bending characteristics:

$$u = Dw, \quad \varphi_x = \frac{\partial u}{\partial x}, \quad \varphi_y = \frac{\partial u}{\partial y}, \quad M_x, \; M_y, \; M_{xy}, \; Q_x, \; Q_y, \; V_x, \; V_y$$

Taking account of the hinge support of the plate along the edges $y = 0, b$, we apply the finite Fourier sine and cosine transforms defined by formulas of the type

$$u^s = \int_0^b Dw(x, y) \sin \beta y \, dy \quad \left(\beta = \frac{n\pi}{b}, \; n = 0, 1, 2, \ldots\right) \tag{4.2}$$

$$Q_y^c = \int_0^b Q_y(x, y) \cos \beta y \, dy, \quad q^s = \int_0^b q(x, y) \sin \beta y \, dy$$

to this system.

Upon performing the integration by parts associated with this operation, the range of integration should be separated into two: $(0, l - 0)$, $(l + 0, b)$, and it should be taken into account that, the angle of rotation φ_y becomes discontinuous with passage through the hinge, i.e.,

$$\varphi_y|_{l-0} - \varphi_y|_{l+0} = \chi(x) \quad (\chi(x) \equiv 0, \; x \notin (c_0, c_1)) \tag{4.3}$$

We will consequently have (we consider the Poisson's ratio zero)

$$\frac{dQ_x^s}{dx} - \beta Q_y^c = -q^s, \quad \frac{dM_x^s}{dx} - \beta M_{xy}^c - Q_x^c = 0$$

$$\frac{dM_{xy}^c}{dx} + \beta M_y^s - Q_y^c = 0, \quad \frac{d\varphi_x^s}{dx} + M_x^s = 0, \quad \frac{du^s}{dx} = \varphi_x^s$$

$$M_y^s - \beta \varphi_y^c + \sin \beta l \chi(x) = 0, \quad M_{xy}^c + \beta \varphi_x^s = 0, \quad \varphi_y^c = \beta u^s$$

$$\frac{dM_{xy}^c}{dx} + Q_y^c = V_y^c, \quad V_x^c = Q_x^s - \beta M_{xy}^c$$

By elimination and linear combinations, the system obtained can be reduced to the differential equation

$$\frac{dz}{dx} + Pz = f \quad (a_0 < x < a_1) \tag{4.4}$$

$$P = \left\| \begin{matrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 2\beta^2 & -1 \\ 0 & 1 & 0 & 0 \\ -\beta^4 & 0 & 0 & 0 \end{matrix} \right\|, \quad f = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ -q^s - \beta^2 \sin \beta l \chi(x) \end{matrix} \right\|$$

in the vector function (column matrix) $z(x) = (u^s, M_x^s, \varphi_x^s, V_x^s)$. If the boundary conditions on the edges $x = a_j$ which have the following form

$$Az(a_0) + Bz(a_1) = \gamma \tag{4.5}$$

in the general case, is added to the equation obtained (A, B are matrices, and γ is a fourth order column matrix), then we arrive at a one-dimensional boundary value problem for the vector z. If the matrix Green's function (the Green's matrix $G(x, \xi)$) is constructed for the semi-homogeneous $(\gamma = 0)$ boundary value problem

(4.4), (4.5) and the basis matrix $\Psi(x)$ satisfying the equation and boundary conditions

$$d\Psi/dx + P\Psi = 0 \tag{4.6}$$

$$A\Psi(a_0) + B\Psi(a_1) = I \tag{4.7}$$

then the solution of the inhomogeneous boundary value problem (4.4),(4.5) can be obtained by means of the formula

$$z(x) = \int_{a_0}^{a_1} G(x, \xi) f(\xi)\, d\xi + \Psi(x)\gamma \tag{4.8}$$

which is the analog of (1.9).

Note. If we do not proceed from a system of first order equations but from the equation of Sophie Germain, then we have one scalar fourth order equation instead of (4.4). However, $\chi''(x)$, with nonintegrable singularities at the points $x = c_j$, will be contained in its right side. Such difficulties do not occur in the solution of different mathematical physics problems (in particular, the plate bending problem), and the method elucidated can be applied directly to the high order governing equations (particularly to the Sophie Germain equation) if this is certainly convenient.

Finding the vector $z(x)$ and using the inversion formula for the finite sine transform, we find $M_y(x, y)$. This permits realization of the condition (4.1), obtaining thereby the integral equation for the desired function $\chi(x)$.

As we see, the principal difficulty (technical) is in constructing the Green's matrix. Since the general method [4] of constructing such matrices is awkward and inconvenient from the viewpoint of extracting the singular parts in the equations obtained, a special method of constructing the Green's matrix is elucidated below.

5. For generality, let us consider the vector $z(x)$ to be of n-th order. A so-called matrizant [8], i.e., the solution $Z(x)$ of (4.6) which possesses the property

$$Z(0) = I \tag{5.1}$$

can be constructed for equations of the type (4.4) with constant coefficients.

Indeed, let us introduce the matrix

$$M(\zeta) = I\zeta + P \tag{5.2}$$

into the considerations, whose determinant will evidently be a polynomial of degree n, i.e.,

$$|M(\zeta)| = Q_n(\zeta) = \prod_{j=0}^{n-1} (\zeta - \zeta_j) \tag{5.3}$$

If ζ does not coincide with any of the roots (they may even be multiple) of this polynomial, then there will exist

$$M^{-1}(\zeta) = \Delta^*(\zeta)\, Q_n^{-1}(\zeta) \tag{5.4}$$

where $\Delta^*(\zeta)$ is the transpose matrix to the matrix of cofactors for the elements of the matrix (5.2).

Let us show that

$$Z(x) = \frac{1}{2\pi i} \int\limits_{C} \frac{\Delta^*(\zeta)}{Q_n(\zeta)} e^{\zeta x}\, d\zeta \tag{5.5}$$

(C is any closed contour enclosing all the zeroes of the polynomial Q_n).

Let us substitute (5.5) into (4.6), taking into account that the matrix (5.4) is inverse to the matrix $M(\zeta)$ and let us use the Cauchy theorem. We consequently arrive at an identity. There remains to show the validity of the equation (5.1) which is equivalent to the following:

$$\frac{1}{2\pi i} \oint\limits_{C} \frac{\Delta^*(\zeta)\, a\zeta}{Q_n(\zeta)} = I \tag{5.6}$$

In order to see its validity, it is sufficient to compute the residue of the integrand (5.6) for $\zeta = \infty$.

The expression (5.5) obtained for the matrizant can evidently be written in the following form:

$$Z(x) = \sum_{j=0}^{n-1} \operatorname{Res} \left[\frac{\Delta^*(\zeta)\, e^{\zeta x}}{Q_n(\zeta)} \right]_{\zeta=\zeta_j} \tag{5.7}$$

Knowing the matrizant, we can construct the basis matrix possessing the property (4.7)

$$\Psi(x) = Z(x)\, C \tag{5.8}$$

It can be shown, exactly as in the case of (1.10), that the following formula for the Green's matrix of the boundary value problem (4.4), (4.5) is valid

$$G(x, \xi) = \Phi(x, \xi) - \Psi(x)\, [A\Phi(a_0, \xi) + B\Phi(a_1, \xi)] \tag{5.9}$$

where $\Phi(x, \xi)$ is the fundamental matrix performing the same role as does the fundamental function in the scalar case.

In the case of (4.4) with constant coefficients, the fundamental matrix dependent on the difference between the arguments is easily constructed if it is taken into account [4] that every matrix $\Phi(x - \xi)$ satisfying the matrix equation (4.6) for $x > \xi$ and for $x < \xi$, and having the jump

$$\Phi(+0) - \Phi(-0) = I \tag{5.10}$$

for $x = \xi$, is fundamental.

The matrix

$$\Phi(y) = \pm \frac{1}{2\pi i} \oint\limits_{C_\pm} \frac{\Delta^*(\zeta)}{Q_n(\zeta)} e^{\zeta y}\, d\zeta \quad (y \gtrless 0) \tag{5.11}$$

for instance, possesses this property, where C_+ is the contour enclosing any m roots of the polynomial $Q_n(\zeta)$, and C_- is the contour enclosing the remaining $n - m$ roots. In order to see this, it is sufficient to take into account that each of the contour integrals in (5.11) satisfies (4.6) on the basis of (5.4). The equality (5.10) turns out to be valid on the basis of (5.6). The fundamental matrix can also be constructed by applying a Fourier transform to (4.4) just as was done in obtaining the fundamental function (2.5). We consequently arrive at the formula

$$\Phi(y)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Delta^*(\zeta)}{Q_n(\zeta)}e^{\zeta y}\,d\zeta \tag{5.12}$$

which is a particular realization of (5.11).

Comparing (5.12), (5.11) with (5.5), we see that the fundamental matrix is expressed in terms of the same residues as the matrizant (5.7). As in the scalar case, it will sometimes be convenient to substitute one of the following expressions:

$$\Phi(x,\xi)=\Phi(x-\xi)\pm\Phi(x+\xi) \tag{5.13}$$

into (5.9) in the presence of fundamental matrices of the type (5.11).

Note. Formula (5.9) is valid even in the general case of (4.4) when a matrix P_0 is in front of the derivative instead of unity matrix ($\det P_0 \neq 0$) and when both matrices P_0 and P_1 depend on the variable x. The method elucidated here to construct the matrizant and the fundamental matrix is based on the Cauchy method [9] and on its development by M.G.Krein in application to an ordinary differential equation with constant coefficients.

6. Let us apply the formula obtained to the solution of the problem posed above on the linear plastic hinge by assuming, for definiteness, that the edges $x=a_0,\ a_1$ of the plate are hinge-supported, where

$$a_0=0,\ a_1=a.$$

When using (5.9) it turns out to be convenient to operate with their blocks rather than with the components of the matrices P, A, B, z, f. This is associated partially with the fact that the boundary conditions (4.5) are usually given in separated (with respect to the edges) form in application to definite sets (blocks) $z^{\pm}(x)$ of elements of the vector functions $z(x)=\{z^+(x),\ z^-(x)\}$. For example, in the case of the problem selected if u^s and $M_x{}^s$ refer to the block $z^+(x)$ and $\varphi_x{}^s,\ V_x{}^s$ are referred to the block $z^-(x)$, then the hinge-support conditions are written in the form

$$z^+(0)=0,\quad z^+(a)=0 \tag{6.1}$$

Let us represent the matrices in (4.4)—(4.7) in the form

$$P=\begin{Vmatrix}0 & P^+\\ P^- & 0\end{Vmatrix},\quad P^+=\begin{Vmatrix}-1 & 0\\ 2\beta^2 & -1\end{Vmatrix},\quad P^-=\begin{Vmatrix}0 & 1\\ -\beta^4 & 0\end{Vmatrix} \tag{6.2}$$

$$f=\begin{Vmatrix}f^+\\ f^-\end{Vmatrix},\quad f^+=\begin{Vmatrix}0\\ 0\end{Vmatrix},\quad f^-=\begin{Vmatrix}0\\ -q^s-\beta^2\sin\beta l\chi(x)\end{Vmatrix}$$

$$A=\begin{Vmatrix}I & 0\\ 0 & 0\end{Vmatrix},\quad B=\begin{Vmatrix}0 & 0\\ I & 0\end{Vmatrix},\quad \gamma=0$$

The blocks of the Green's matrix $G(x,\xi)$ and the basis matrix $\Psi(x)$ will be marked by superscripts, i.e. $G^{jk}(x,\xi),\ \Psi^{jk}(x),\ j,k=0,1.$ On the basis of

(4.8) and (6.2) we will hence have

$$z^{+}(x)=\left\|\begin{matrix}u^{s}\\ M_{x}^{\ s}\end{matrix}\right\|=\int_{0}^{a}G^{01}(x,\xi)f^{-}(\xi)\,d\xi \tag{6.3}$$

i.e., only one block of the Green's matrix is needed to solve the problem selected.

On the basis of (6.2), the matrix (5.2) is written in the block form

$$M(\zeta)=\left\|\begin{matrix}\zeta I & P^{+}\\ P^{-} & \zeta I\end{matrix}\right\| \tag{6.4}$$

We denote the block of its inverse matrix by M_{-1}^{jk}. It is convenient to find them from the four equations obtained from the matrix equality

$$\left\|\begin{matrix}\zeta I & P^{+}\\ P^{-} & \zeta I\end{matrix}\right\|\left\|\begin{matrix}M_{-1}^{\ 00} & M_{-1}^{\ 01}\\ M_{-1}^{\ 10} & M_{-1}^{\ 11}\end{matrix}\right\|=\left\|\begin{matrix}I & 0\\ 0 & I\end{matrix}\right\|$$

We consequently obtain

$$Q_{n}(\zeta)=(\zeta^{2}-\beta^{2})^{2},\quad \Delta^{*}(\zeta)=\sum_{j=0}^{3}\zeta^{j}\Delta_{j} \tag{6.5}$$

$$\Delta_{0}=\left\|\begin{matrix}0 & S^{+}\\ S^{-} & 0\end{matrix}\right\|,\quad \Delta_{1}=-\left\|\begin{matrix}R^{+} & 0\\ 0 & R^{-}\end{matrix}\right\|,\quad \Delta_{2}=-\left\|\begin{matrix}0 & P^{+}\\ P^{-} & 0\end{matrix}\right\|$$

$$\Delta_{3}=I,\quad R^{+}=\left\|\begin{matrix}2\beta^{2} & 1\\ -\beta^{4} & 0\end{matrix}\right\|,\quad R^{-}=\left\|\begin{matrix}0 & 1\\ -\beta^{4} & 2\beta^{2}\end{matrix}\right\|$$

$$S^{+}=R^{+}P^{+}=\left\|\begin{matrix}0 & -1\\ \beta^{4} & 0\end{matrix}\right\|,\quad S^{-}=R^{-}P^{-}=\left\|\begin{matrix}-\beta^{4} & 0\\ -2\beta^{6} & -\beta^{4}\end{matrix}\right\|$$

On the basis of (6.5), the formula (5.7) for the matrizant reduces to

$$Z(x)=\sum_{j=0}^{3}\Delta_{j}u^{(j)}(x),\quad u(x)=\frac{\beta x\operatorname{ch}\beta x-\operatorname{sh}\beta x}{2\beta^{3}},\quad u^{(j)}=\frac{d^{j}u}{dx^{j}} \tag{6.6}$$

and after evaluation of the residues (5.12) acquires the form

$$\Phi(y)=\sum_{j=0}^{3}\Delta_{j}\varphi^{(j)}(y),\quad \varphi(y)=\frac{e^{-\beta|y|}(1+\beta|y|)}{4\beta^{3}} \tag{6.7}$$

To construct the basis matrix, let us write the boundary conditions (4.7) in blocks

$$\Psi^{00}(0)=\Psi^{01}(a)=I,\quad \Psi^{01}(0)=\Psi^{00}(a)=0 \tag{6.8}$$

The first and third equalities will be satisfied by virtues of (5.1) if we set

$$C=\left\|\begin{matrix}I & 0\\ C^{10} & C^{11}\end{matrix}\right\| \tag{6.9}$$

in (5.8). The realization of the remaining two will result in the formulas

$$C^{11}=[Z^{01}(a)]^{-1}=N,\quad C^{10}=-NZ^{00}(a)=-N' \tag{6.10}$$

Taking (6.6) into account, we find $(\rho = a\beta)$

$$N = \frac{1}{2\beta \operatorname{sh}^2 \rho} \begin{Vmatrix} \beta^2 (\rho \operatorname{ch} \rho + \operatorname{sh} \rho) & \rho \operatorname{ch} \rho - \operatorname{sh} \rho \\ \beta^4 (3 \operatorname{sh} \rho + \rho \operatorname{ch} \rho) & \beta^2 (\rho \operatorname{ch} \rho + \operatorname{sh} \rho) \end{Vmatrix} \tag{6.11}$$

$$N' = \frac{1}{4\beta \operatorname{sh}^2 \rho} \begin{Vmatrix} \beta^2 (2\rho + \operatorname{sh} 2\rho) & 2\rho - \operatorname{sh} 2\rho \\ \beta^4 (3 \operatorname{sh} 2\rho + 2\rho) & \beta^2 (2\rho + \operatorname{sh} 2\rho) \end{Vmatrix}$$

Therefore, the required basis function is determined by using (5.8) and (6.9) — (6.11) by the formula

$$\Psi(x) = \sum_{j=0}^{3} u^{(j)}(x) \Delta_j \begin{Vmatrix} I & 0 \\ -N' & N \end{Vmatrix} \tag{6.12}$$

Taking the second expression from (5.13) and using (6.7) and (6.12), on the basis of (5.9) we obtain the Green's matrix for which the block needed has the form

$$\begin{aligned} G^{01}(x, \xi) = {} & [\varphi(x - \xi) - \varphi(x + \xi)] S^+ - [\varphi^{(2)}(x - \xi) - \varphi^{(2)}(x + \xi)] \times \\ & P^+ - \frac{u(x)}{2\beta^3 e^\rho} [\operatorname{sh} \beta\xi S^+ N E^+ + (\rho \operatorname{sh} \beta\xi - \beta\xi \operatorname{ch} \beta\xi) S^+ N E^-] + \\ & \frac{u^{(2)}(x)}{2\beta^3 e^\rho} [\operatorname{sh} \beta\xi P^+ N E^+ + (\rho \operatorname{sh} \beta\xi - \beta\xi \operatorname{ch} \beta\xi) P^+ N E^-] \end{aligned} \tag{6.13}$$

$$(\beta a = \rho, \; E^{\mp} = S^+ \mp \beta^2 P^+)$$

Marking the components of the block obtained by subscripts, we have on the basis of (6.3), (6.2) and (6.5), (6.11)

$$u^s(x) = -\int_{c_0}^{c_1} G_{01}{}^{01}(x, \xi) \beta^2 \sin \beta l \chi(\xi) \, d\xi - \int_0^a G_{01}{}^{01}(x, \xi) q^s(\xi) \, d\xi \tag{6.14}$$

$$G_{01}{}^{01}(x, \xi) = \varphi(x + \xi) - \varphi(x - \xi) - k(x, \xi, \beta)$$

$$\begin{aligned} k(x, \xi, \beta) = {} & [2\beta^3 e^\rho \operatorname{sh} \rho]^{-1} [\beta x \operatorname{ch} \beta x \operatorname{sh} \beta\xi + \beta\xi \operatorname{sh} \beta x \operatorname{ch} \beta\xi - \\ & \operatorname{sh} \beta x \operatorname{sh} \beta\xi (1 + \rho e^\rho \operatorname{cosech} \rho)] \end{aligned}$$

Hence, by using the inversion formula for a finite sine transform, we find the deflection of the plate and the bending moment. After some manipulation, the next realization of condition (4.1) results in the following singular integral equation:

$$\frac{3}{4\pi} \frac{d^2}{dx^2} \int_{c_0}^{c_1} \ln \operatorname{cth} \frac{\pi}{2b} |x - \xi| \chi(\xi) \, d\xi + \tag{6.15}$$

$$\int_{c_0}^{c_1} K_p(x, \xi) \chi(\xi) \, d\xi = M_p - Q(x) \quad (c_0 \leqslant x \leqslant c_1)$$

The regular part of the kernel $K_p(x, \xi)$ and the function $Q(x)$ are determined by the formulas

$$K_p(x, \xi) = \frac{1}{4b} \frac{d^2}{dx^2} \frac{x - \xi}{\sin \pi b^{-1}(x - \xi)} + K_4\left(x, \xi, \frac{b}{2}, \frac{b}{2}\right)$$

$$Q(x) = \int_0^a \int_0^b \left[S_2\left(x-\xi, \frac{b}{2}, \eta\right) + K_2\left(x, \xi, \frac{b}{2}, \eta\right)\right] q(\xi, \eta)\, d\xi\, d\eta$$

$$\begin{Bmatrix} K_j(x, \xi, y, \eta) \\ S_j(X, y, \eta) \end{Bmatrix} = \frac{2}{b} \sum_{n=1}^{\infty} \begin{Bmatrix} k(x, \xi, \beta) - \varphi(x+\xi) \\ \varphi(X) \end{Bmatrix} \beta^j \sin \beta y \sin \beta \eta$$

$$(\beta = n\pi b^{-1},\ j = 0, 1 \ldots)$$

Finding the (approximate) solution of the equation obtained by orthogonal poly-nomials, say, by using the expansion [10]

$$\ln \operatorname{cth} |y| = \ln \frac{1}{|y|} + \sum_{k=1}^{\infty} \frac{B_{2k}(2^{2k-1}-1)2^{2k}}{k(2k)!} y^{2k}$$

and the spectral relationship [11]

$$\frac{1}{\pi} \frac{d^2}{dx^2} \int_{-1}^{1} \ln \frac{1}{|x-\xi|} \sqrt{1-\xi^2}\, U_n(\xi)\, d\xi = -(n+1) U_n(x) \quad (|x| \leqslant 1)$$

we obtain the transform of the deflections by (6.14) and the plate deflections and de-sign forces by using it.

In conclusion, let us note that the more complex problem of a cruciform plastic hinge can be considered and reduced to a system of two singular integral equations, and also problems in the presence of analogous open slots and narrow elastic beam rein-forcements. In general the method elucidated is applicable to all those boundary value problems with slits and inclusions which are solvable by the method of integral trans-forms upon removal of the defects mentioned. It is only important that the slits and inclusions be inscribed in an appropriate coordinate grid.

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